Published by Institute of Physics Publishing for SISSA/ISAS

Received: *August 31, 2004*
Revised: *October 28, 2004*
Accepted: *October 28, 2004*

Superconformal Selfdual σ-Models

Louis Paulot

Physique théorique et mathématique and International Solvay Institutes
Université libre de Bruxelles
C.P. 231, B-1050 Bruxelles, Belgium
E-mail: lpaulot@ulb.ac.be

Abstract: A range of bosonic models can be expressed as (sometimes generalized) σ-models, with equations of motion coming from a selfduality constraint. We show that in $D = 2$, this is easily extended to supersymmetric cases, in a superspace approach. In particular, we find that the configurations of fields of a superconformal $\mathfrak{G}/\mathfrak{H}$ coset models which satisfy some selfduality constraint are automatically solutions to the equations of motion of the model. Finally, we show that symmetric space σ-models can be seen as infinite-dimensional $\widetilde{\mathfrak{G}}/\widetilde{\mathfrak{H}}$ models constrained by a selfduality equation, with $\widetilde{\mathfrak{G}}$ the loop extension of $\mathfrak{G}$ and $\widetilde{\mathfrak{H}}$ a maximal subgroup. It ensures that these models have a hidden global $\widetilde{\mathfrak{G}}$ symmetry together with a local $\widetilde{\mathfrak{H}}$ gauge symmetry.

Keywords: Gauge Symmetry, Global Symmetries, Supersymmetry and Duality, Integrable Equations in Physics.

Contents

JHEP10(2004)071

1. Introduction

Several (classical) bosonic models have been expressed as generalized σ-models constrained by a selfduality equation. In this formalism, the set of fields is doubled and equations of motion are given by Bianchi identities of dual fields. This class of models contains the bosonic matter sector of supergravities and especially eleven-dimensional supergravity and reductions of these on tori, up to $D = 3$ [1]–[6], as well as all models obtained by oxidation of $D = 3$ principal σ-models for all simple groups [7]–[9], and is closely related to the so-called "E_{11} conjecture" [4, 10].

The inclusion of the fermionic and gravity sectors of such models into this formalism has not yet been achieved, although there are some results on $D = 2$ gravity models coming from $D = 3$ coset models [11]–[15], sometimes including fermions in the case of $N = 16$ supergravity [16]–[19]. These $D = 2$ models can indeed be seen as integrable systems, which opens a path to quantization [20]–[23]. One can also find attempts of including fermions in the doubled formalism for eleven-dimensionnal and type IIA supergravities in [24, 25].

In a program of including fermions, gravity and supersymmetry into such a formalism for generalized σ-models, especially in higher dimension, and revealing hidden symmetries of theories, we begin here by studying $D = 2$ models without (super)gravity. We extend selfduality of σ-models to superconformal cases, by defining a Hodge star on superspace that we use together with a pseudo-involution. Special cases have already been studied in the context of solitonic and instantonic solutions of some supersymmetric σ-models, starting from [26]–[28]. Here we develop a general theory where the target-space can be any Lie group or symmetric space.

After recalling basic facts on the selfdual formalism in section 2, we introduce its superconformal version (section 3); we study then solutions to a selfduality constraint in a superconformal symmetric space σ-model in section 4, and we show that selfdual sets of fields are special solutions to the usual equations of motion. As a simple example, we treat very explicitely the case of $\mathrm{SL}(2)/\,\mathrm{SO}(2)$.

Our main result is in section 5: we give a formulation of such $\mathfrak{G}/\mathfrak{H}$ σ-models as selfdual infinite $\widetilde{\mathfrak{G}}/\widetilde{\mathfrak{H}}$ σ-models, where $\widetilde{\mathfrak{G}}$ is the loop extension of the group $\mathfrak{G}$, and $\widetilde{\mathfrak{H}}$ a maximal subgroup containing $\mathfrak{H}$. Physical fields with their Bianchi identities and equations of motion appear through the Lax pair associated to the model. The hidden group of symmetries of such models, including non-local ones, have been discussed for some time [29]–[36] with different conclusions. With our construction, it is manifest that in addition to (super)conformal symmetry on the worldsheet, there is a global symmetry $\widetilde{\mathfrak{G}}$ and a local gauge symmetry $\widetilde{\mathfrak{H}}$. It has been checked in some cases that such symmetries survive quantization ([37] and references therein).

2. Selfdual σ-models

2.1 σ-models

A σ-model is described by a map ϕ from spacetime Σ to a target manifold $\mathcal{M}$. Let n be the dimension of the target space.

Here we specify to the case where $\mathcal{M}$ is a simply connected Lie group $\mathfrak{G}$. Let $\{T_i\}$ be a basis of generators of the tangent Lie algebra $\mathfrak{g}$. Elements of $\mathfrak{G}$ can be parametrized locally, in the vicinity of $\mathcal{V}_0$, as

$$\mathcal{V} = \exp\left(\sum_i \phi^i T_i\right)\mathcal{V}_0 \tag{2.1}$$

with n scalars ϕ^i, where n is the dimension of $\mathfrak{G}$.

On $\mathcal{M} = \mathfrak{G}$, we have the $\mathfrak{g}$-valued Maurer-Cartan form

$$\sigma = \partial_i \mathcal{V}\,\mathcal{V}^{-1}\mathrm{d}\phi^i, \tag{2.2}$$

JHEP10(2004)071

where $\partial_i = \frac{\partial}{\partial \phi^i}$. We can compute it with help of the Baker-Campbell-Hausdorff formula

$$\partial_i \mathcal{V}\,\mathcal{V}^{-1} = \partial_i e^{\phi}\, e^{-\phi} = \partial_i \phi + \frac{1}{2!}[\phi, \partial_i \phi] + \frac{1}{3!}[\phi, [\phi, \partial_i \phi]] + \cdots \tag{2.3}$$

(where $\phi = \sum \phi^j T_j$).

We define the field strength associated to ϕ as the pullback on Σ of the Maurer-Cartan form on $\mathfrak{G}$.

$$\mathcal{G} = \mathrm{d}\mathcal{V}\,\mathcal{V}^{-1} = \partial_\mu \phi^i\, \partial_i \mathcal{V}\,\mathcal{V}^{-1} \mathrm{d}x^\mu, \tag{2.4}$$

where ∂_i is the target-space derivative and ∂_μ the spacetime derivative, with respect to x^μ.

This field strength is invariant under action of $\mathfrak{G}$. Indeed, if one acts on $\mathcal{V}$:

$$\mathcal{V} \longrightarrow \mathcal{V}\Lambda \tag{2.5}$$

with elements

$$\Lambda = e^{(\lambda^i T_i)} \tag{2.6}$$

defined with closed scalars λ^i ($\mathrm{d}\lambda^i = 0$), $\mathcal{G}$ becomes

$$\mathcal{G} \longrightarrow \mathrm{d}\mathcal{V}\Lambda\Lambda^{-1}\mathcal{V}^{-1} + \mathcal{V}\mathrm{d}\Lambda\Lambda^{-1}\mathcal{V}^{-1}, \tag{2.7}$$

where the second term vanishes because of $\mathrm{d}\lambda = 0$.

The σ-model structure we have recalled in this part is valid for any number of spacetime dimensions. We restrict now to two dimensions for the rest of the paper. Moreover, we will work in Euclidean signature. We will indicate how to transpose to lorentzian signature.

In euclidean signature,[1] it will be convenient to use complex notations and to write the field strength as

$$\begin{aligned} \mathcal{G} &= G_z \mathrm{d}z + G_{\bar{z}} \mathrm{d}\bar{z} \\ &= \partial_z \phi^i\, \partial_i \mathcal{V}\,\mathcal{V}^{-1} \mathrm{d}z + \partial_{\bar{z}} \phi^i\, \partial_i \mathcal{V}\,\mathcal{V}^{-1} \mathrm{d}\bar{z}. \end{aligned} \tag{2.8}$$

In order to make the contact with the supersymmetric generalization easier, we also think of this 1-form field as a vector in the cotangent bundle:

$$\mathcal{G} = \begin{pmatrix} G_z \\ G_{\bar{z}} \end{pmatrix}. \tag{2.9}$$

2.2 Selfduality

We consider the group element $\mathcal{V}$ as a *doubled* set of fields: equations of motion are given by a selfduality equation.

Namely, let $\mathcal{S}$ be a pseudo involution of $\mathfrak{g}$, exchanging T_a's, such that

$$*^2 \mathcal{S}^2 = 1. \tag{2.10}$$

If we choose a euclidean signature, we have $*^2 = -1$ for 1-forms, and therefore $\mathcal{S}^2 = -1$. For a lorentzian signature, it would be $*^2 = 1$ for 1-forms, and $\mathcal{S}^2 = 1$.

[1] In a lorentzian setting, we would use lightcone coordinates x^+ and x^-.

Thus we can write a selfduality equation

$$*\mathcal{S}\mathcal{G} = \mathcal{G}, \tag{2.11}$$

where $*$ acts on the differentials $\mathcal{G}^a$ and $\mathcal{S}$ on generators T_a.

Following from the definition of $\mathcal{G}$, we have the Bianchi identity

$$\mathrm{d}\mathcal{G} - \mathcal{G} \wedge \mathcal{G} = 0. \tag{2.12}$$

If we use the selfduality relation (2.11) to reduce by one half the number of fields, one half of the set of Bianchi identities becomes the equations of motion of the physical fields. Indeed we have

$$\mathrm{d}*\mathcal{S}\mathcal{G} - \mathcal{G} \wedge \mathcal{G} = 0. \tag{2.13}$$

A trivial example is the free scalar φ. We take an abelian algebra of dimension two, with generators h_1 and h_2. Let

$$\phi = \phi_1 h_1 + \phi_2 h_2; \tag{2.14}$$

the curvature is

$$\mathcal{G} = \mathrm{d}\phi_1\, h_1 + \mathrm{d}\phi_2\, h_2. \tag{2.15}$$

With $\mathcal{S}$ exchanging h_1 and h_2,

$$\begin{aligned} \mathcal{S}h_1 &= h_2 \\ \mathcal{S}h_2 &= -h_1, \end{aligned} \tag{2.16}$$

the second Bianchi identity

$$\mathrm{d}\mathrm{d}\phi_2 = 0 \tag{2.17}$$

becomes the equation of motion of ϕ_1:

$$\mathrm{d}*\mathrm{d}\phi_1 = 0. \tag{2.18}$$

In complex notation, in the conformal gauge, the Hodge star $*$ has a very simple expression, which will be convenient for the supersymmetric generalization:

$$*(G_z \mathrm{d}z + G_{\bar{z}} \mathrm{d}\bar{z}) = -iG_z \mathrm{d}z + iG_{\bar{z}} \mathrm{d}\bar{z}. \tag{2.19}$$

We remark that the selfduality equation is conformally invariant, so that the model has (spacetime) conformal symmetry, with G_z of weight $(1,0)$ and $G_{\bar{z}}$ of weight $(0,1)$. Indeed, we have

$$G_{z'} = \partial_{z'}\mathcal{V}\,\mathcal{V}^{-1} = (\partial_{z'}z)\partial_z\mathcal{V}\,\mathcal{V}^{-1} = (\partial_{z'}z)G_z, \tag{2.20}$$

with the analougous for $G_{\bar{z}}$.

JHEP10(2004)071

3. Supersymmetric version

3.1 $N = (1,1)$ extension

We work now in $D = 2$ $N = (1,1)$ superspace: we have two odd variables θ and $\bar{\theta}$. Supersymmetry generators are

$$\begin{aligned} Q &= \partial_\theta - \theta\partial_z \\ \overline{Q} &= \partial_{\bar{\theta}} - \bar{\theta}\partial_{\bar{z}} \end{aligned} \tag{3.1}$$

and covariant derivatives

$$\begin{aligned} D &= \partial_\theta + \theta\partial_z \\ \overline{D} &= \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}. \end{aligned} \tag{3.2}$$

We construct a super-σ-model with a scalar superfield $\mathcal{V}$ with values in $\mathfrak{G}$

$$\mathcal{V} = e^{\Phi}\,\mathcal{V}_0, \tag{3.3}$$

where

$$\Phi = \sum_i \Phi^i T_i \tag{3.4}$$

is a scalar superfield with values in $\mathfrak{g}$.

The field strength is defined as:

$$\mathcal{G} = \begin{pmatrix} G \\ \overline{G} \end{pmatrix} = \begin{pmatrix} D\mathcal{V}\,\mathcal{V}^{-1} \\ \overline{D}\mathcal{V}\,\mathcal{V}^{-1} \end{pmatrix}. \tag{3.5}$$

G and $\overline{G}$ can be computed from Φ with help of the Baker-Campbell-Hausdorff formula:

$$G = De^{\Phi}\,e^{-\Phi} = D\Phi + \frac{1}{2!}[\Phi, D\Phi] + \frac{1}{3!}[\Phi,[\Phi,D\Phi]] + \cdots \tag{3.6}$$

and the analogous for $\overline{G}$ with $\overline{D}$.

A Bianchi identity follows from the definition of $\mathcal{G}$:

$$\overline{D}G + D\overline{G} = \left[G, \overline{G}\right]. \tag{3.7}$$

As G and $\overline{G}$ are odd superfields, the bracket is in fact an anticommutator.

We define a "Hodge star" by

$$*\mathcal{G} = * \begin{pmatrix} G \\ \overline{G} \end{pmatrix} = \begin{pmatrix} -i\,G \\ i\,\overline{G} \end{pmatrix}. \tag{3.8}$$

Together with the pseudo-involution $\mathcal{S}$ defined as in the purely bosonic case, we can write a selfduality equation

$$*\mathcal{S}\mathcal{G} = \mathcal{G}, \tag{3.9}$$

which reduces the number of physical fields by one half and turns one half of Bianchi identities into equations of motion:

$$-i\overline{D}(\mathcal{S}G) + iD(\mathcal{S}\overline{G}) = \left[G, \overline{G}\right]. \tag{3.10}$$

JHEP10(2004)071

This model is invariant under a global $\mathfrak{G}$. We act on $\mathcal{V}$ by $\Lambda = e^{\lambda}$,

$$\mathcal{V} \longrightarrow \mathcal{V}\Lambda, \tag{3.11}$$

with λ a $\mathfrak{g}$-valued superfield. $\mathcal{G}$ is invariant provided

$$\begin{aligned} D\lambda &= 0 \\ \overline{D}\lambda &= 0, \end{aligned} \tag{3.12}$$

i.e. if and only if λ is a constant field: Λ is an element of $\mathfrak{G}$.

As all operators we use are covariant with respect to supersymmetry generators Q and $\overline{Q}$, we get a manifestly supersymmetric theory. Moreover, it is clear that the selfduality equation is superconformally invariant, with G and $\overline{G}$ of respective weights $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. Thus the selfdual σ-model we have defined have superconformal symmetry in addition to its $\mathfrak{G}$ target-space symmetry.

We want to show now that, if we truncate the superfield

$$\Phi = \phi + \theta\psi + \bar{\theta}\widetilde{\psi} + \theta\bar{\theta}F \tag{3.13}$$

to its first component

$$\phi = \sum_i \phi^i T_i, \tag{3.14}$$

we recover precisely the bosonic selfdual sigma model defined in section 2.

Indeed, we get

$$G = De^{\phi}\, e^{-\phi} = \theta\partial_z\phi + \frac{1}{2!}[\phi, \theta\partial_z\phi] + \frac{1}{3!}[\phi, [\phi, \theta\partial_z\phi]] + \cdots = \theta\, G_z^{(0)} \tag{3.15}$$

and

$$\overline{G} = \overline{D}e^{\phi}\, e^{-\phi} = \bar{\theta}\partial_{\bar{z}}\phi + \frac{1}{2!}[\phi, \bar{\theta}\partial_{\bar{z}}\phi] + \frac{1}{3!}[\phi, [\phi, \bar{\theta}\partial_{\bar{z}}\phi]] + \cdots = \bar{\theta}\, G_{\bar{z}}^{(0)}, \tag{3.16}$$

where we denote by $\mathcal{G}^{(0)} = G_z^{(0)}\mathrm{d}z + G_{\bar{z}}^{(0)}\mathrm{d}\bar{z}$ the bosonic field strength derived from ϕ:

$$\mathcal{G}^{(0)} = \mathrm{d}\left(e^{\phi}\right) e^{-\phi}. \tag{3.17}$$

The Bianchi identity becomes

$$-\theta\bar{\theta}\partial_{\bar{z}}G_z^{(0)} + \theta\bar{\theta}\partial_z G_{\bar{z}}^{(0)} = \theta\bar{\theta}\left[G_z^{(0)}, G_{\bar{z}}^{(0)}\right], \tag{3.18}$$

which can be written for the bosonic field strength $\mathcal{G}^{(0)}$ as

$$\mathrm{d}\mathcal{G}^{(0)} - \mathcal{G}^{(0)} \wedge \mathcal{G}^{(0)} = 0. \tag{3.19}$$

The Hodge star becomes

$$* \begin{pmatrix} \theta G_z^{(0)} \\ \bar{\theta} G_{\bar{z}}^{(0)} \end{pmatrix} = \begin{pmatrix} -i\theta G_z^{(0)} \\ i\bar{\theta} G_{\bar{z}}^{(0)} \end{pmatrix} : \tag{3.20}$$

it reduces to the usual Hodge star in conformal gauge for $\mathcal{G}^{(0)}$.

As a consequence, the self duality equation

$$*\mathcal{S}\mathcal{G} = \mathcal{G} \tag{3.21}$$

gives for the truncated superfield $\Phi = \phi$ the same equations of motion as in the purely bosonic case of section 2. In other words, the selfdual supersymmetric σ-model we have defined in this section is indeed a supersymmetric extension of the bosonic one.

JHEP10(2004)071

3.2 Free scalar superfield and T-duality

Going back to our trivial example, an abelian algebra with two generators h_1 and h_2, we have

$$\mathcal{V} = e^{\Phi} \tag{3.22}$$

with[2]

$$\Phi = -i\Phi_1 h_1 + \Phi_2 h_2, \tag{3.23}$$

which gives a field strength

$$\mathcal{G} = -i\mathcal{G}_1 h_1 + \mathcal{G}_2 h_2 = \begin{pmatrix} -iD\Phi_1 h_1 + D\Phi_2 h_2 \\ -i\overline{D}\Phi_1 h_1 + \overline{D}\Phi_2 h_2 \end{pmatrix}. \tag{3.24}$$

The Bianchi identities

$$\begin{aligned} D\overline{G}_1 + \overline{D}G_1 &= 0 \\ D\overline{G}_2 + \overline{D}G_2 &= 0 \end{aligned} \tag{3.25}$$

are trivial as the algebra is abelian and are simply $[D, \overline{D}] = 0$ applied to Φ_1 and Φ_2.

With $\mathcal{S}$ exchanging h_1 and h_2 as above, the selfduality equation is

$$\begin{aligned} G_1 &= G_2 \\ -\overline{G}_1 &= \overline{G}_2. \end{aligned} \tag{3.26}$$

Thus, the Bianchi identity for $\mathcal{G}_2$ becomes the equation of motion of $\mathcal{G}_1$:

$$D\overline{G}_1 - \overline{D}G_1 = 0, \tag{3.27}$$

that is

$$D\overline{D}\Phi_1 = 0, \tag{3.28}$$

which describes the motion of a free massless scalar superfield. If we write in components

$$\Phi_1 = \phi_1 + \theta\psi_1 + \bar{\theta}\widetilde{\psi}_1 + \theta\bar{\theta}F_1, \tag{3.29}$$

we get the usual equations

$$\partial_z \partial_{\bar{z}} \phi_1 = 0 \tag{3.30}$$

$$\partial_{\bar{z}} \psi_1 = 0 \tag{3.31}$$

$$\partial_z \widetilde{\psi}_1 = 0 \tag{3.32}$$

$$F_1 = 0. \tag{3.33}$$

To emphasize the physical meaning of selfduality, let us write it in term of Φ_i's:

$$\begin{aligned} D\Phi_1 &= D\Phi_2 \\ -\overline{D}\Phi_1 &= \overline{D}\Phi_2. \end{aligned} \tag{3.34}$$

[2] There is a $-i$ factor in the fields definition in order to simplify formulae below. In lorentzian signature, there would not be such i factors everywhere.

With see that we can write Φ_1 and Φ_2 as

$$\begin{aligned} \Phi_1 &= \Phi_c + \Phi_{ac} \\ \Phi_2 &= \Phi_c - \Phi_{ac} \end{aligned} \tag{3.35}$$

in terms of a chiral superfield Φ_c and an antichiral Φ_{ac}. It makes apparent that the duality involved here is simply *T-duality* of a one-dimensional target-space in the worldsheet perspective. It means that the equations of motion follow from the existence of T-duality written as in (3.34). With more flat target-space dimensions, equations of motion are equivalent to the possibility of T-dualize all dimensions.

3.3 More supersymmetries

Generalization to a larger number of supercharges is straightforward. As we consider massless scalars, the $(N, \widetilde{N})$ supersymmetry algebra is

$$\begin{aligned} \left[Q^I, Q^{I'}\right] &= -\delta^{II'}\partial_z \\ \left[\overline{Q}^J, \overline{Q}^{J'}\right] &= -\delta^{JJ'}\partial_{\bar{z}} \\ \left[Q^I, \overline{Q}^J\right] &= 0. \end{aligned} \tag{3.36}$$

In superspace, we have odd variables θ^I and $\bar{\theta}^J$, $I = 1 \dots N$, $J = 1 \dots \widetilde{N}$. The covariant derivatives are

$$\begin{aligned} D^I &= \partial_{\theta^I} + \theta^I \partial_z \\ \overline{D}^J &= \partial_{\bar{\theta}^J} + \bar{\theta}^J \partial_{\bar{z}}. \end{aligned} \tag{3.37}$$

The field strength $\mathcal{G}$ has N left and $\widetilde{N}$ right components

$$\begin{aligned} G^I &= D^I \mathcal{V} \mathcal{V}^{-1} \\ \overline{G}^J &= \overline{D}^J \mathcal{V} \mathcal{V}^{-1} \end{aligned} \tag{3.38}$$

with Bianchi identities

$$\overline{D}^J G^I + D^I \overline{G}^J = \left[G^I, \overline{G}^J\right]. \tag{3.39}$$

The Hodge star $*$ acts for all values of indices as

$$\begin{aligned} G^I &\longrightarrow -i\, G^I \\ \overline{G}^J &\longrightarrow i\, \overline{G}^J, \end{aligned} \tag{3.40}$$

such that the selfduality equation $*\mathcal{S}\mathcal{G} = \mathcal{G}$ reads

$$\begin{aligned} -i\mathcal{S}G^I &= G^I \\ i\mathcal{S}\overline{G}^J &= \overline{G}^J. \end{aligned} \tag{3.41}$$

It is essential that the sign is the same for all values of the indices: one can change a global sign in $\mathcal{S}$ but left and right components must take well defined opposite signs.

Equations of motion are obtained by introducing this selfduality equation into the Bianchi identities:

$$-i\overline{D}^J\mathcal{S}G^I + iD^I\mathcal{S}\overline{G}^J = \left[G^I, \overline{G}^J\right]. \tag{3.42}$$

If we truncate a $(N, \widetilde{N})$ superfield Φ to its lower component ϕ, all of these equations of motion give back the equation of the bosonic selfdual σ-model, exactly as in section 3.1.

3.4 $(N, 0)$ supersymmetry

The $(N, 0)$-superconformal case is slightly different: one has covariant D^I in the left sector and the usual derivative $\overline{\partial}$ in the right one.

The field strength $\mathcal{G}$ has now N fermionic components on the left and one bosonic component on the right:

$$\begin{aligned} G^I &= D^I\mathcal{V}\,\mathcal{V}^{-1} \\ G_{\bar{z}} &= \partial_{\bar{z}}\mathcal{V}\,\mathcal{V}^{-1}. \end{aligned} \tag{3.43}$$

The Bianchi identities now involve commutators:

$$D^I G_{\bar{z}} - \partial_{\bar{z}}G^I = \left[G^I, G_{\bar{z}}\right]. \tag{3.44}$$

The Hodge star $*$ obviously acts on $\mathcal{G}$ as

$$\begin{aligned} G^I &\longrightarrow -i\,G^I \\ G_{\bar{z}} &\longrightarrow i\,G_{\bar{z}}, \end{aligned} \tag{3.45}$$

and the selfduality equation $*\,\mathcal{S}\mathcal{G} = \mathcal{G}$ reads

$$\begin{aligned} \mathcal{S}G^I &= G^I \\ -\mathcal{S}G_{\bar{z}} &= G_{\bar{z}}, \end{aligned} \tag{3.46}$$

giving the equations of motion

$$iD^I\mathcal{S}G_{\bar{z}} + i\partial_{\bar{z}}\mathcal{S}G^I = \left[G^I, G_{\bar{z}}\right]. \tag{3.47}$$

If we truncate the $\mathfrak{G}$-valued superfield $\mathcal{V} = e^{\Phi}$ to its lowest component e^{ϕ}, we get[3]

$$\begin{aligned} G^I &= D^I e^{\phi}\, e^{-\phi} = \theta^I\partial_z\phi + \frac{1}{2!}[\phi, \theta^I\partial_z\phi] + \frac{1}{3!}[\phi, [\phi, \theta^I\partial_z\phi]] + \cdots = \theta^I\, G_z^{(0)} \\ G_{\bar{z}} &= \partial_{\bar{z}}e^{\phi}\, e^{-\phi} = \partial_{\bar{z}}\phi + \frac{1}{2!}[\phi, \partial_{\bar{z}}\phi] + \frac{1}{3!}[\phi, [\phi, \partial_{\bar{z}}\phi]] + \cdots = G_{\bar{z}}^{(0)} \end{aligned} \tag{3.48}$$

and the Bianchi identity can be written as

$$\theta^I\partial_z G_{\bar{z}}^{(0)} - \theta^I\partial_{\bar{z}}G_z^{(0)} = \theta^I\left[G_z^{(0)}, G_{\bar{z}}^{(0)}\right], \tag{3.49}$$

which is the Bianchi identity of the bosonic σ-model. The Hodge star gives naturally the usual one on the complex plane, so that we recover the bosonic selfdual σ-model with this truncation of the $(N, 0)$ theory. It is easy to see that all superconformal models defined here contained those with less supersymmetry as truncations, as one expects. The only condition is, trivially, to keep the same algebra $\mathfrak{g}$.

[3]Remember $\mathcal{G}^{(0)} = G_z^{(0)}\mathrm{d}z + G_{\bar{z}}^{(0)}\mathrm{d}\bar{z}$ is the bosonic field strength derived from ϕ.

JHEP10(2004)071

4. Selfduality in symmetric spaces

4.1 Symmetric spaces

There is a class of models which are of great interest, when $\mathcal{V}$ lives in the symmetric space $\mathfrak{G}/\mathfrak{H}$. $\mathfrak{H}$ is a maximal subgroup of $\mathfrak{G}$, defined as the set of fixed points of an involution τ acting on $\mathfrak{G}$. It induces an involution on the tangent algebra $\mathfrak{g}$, which we still denote by τ, and which gives a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}, \tag{4.1}$$

where $\mathfrak{h}$ and $\mathfrak{h}^{\perp}$ are sets of respectively invariant and antiinvariant elements of $\mathfrak{g}$ under the action of τ. We have the maximality property

$$\begin{aligned} [\mathfrak{h},\mathfrak{h}] &\subset \mathfrak{h} \\ \left[\mathfrak{h},\mathfrak{h}^{\perp}\right] &\subset \mathfrak{h}^{\perp} \\ \left[\mathfrak{h}^{\perp},\mathfrak{h}^{\perp}\right] &\subset \mathfrak{h}. \end{aligned} \tag{4.2}$$

A special case of physical interest occurs when $\mathfrak{H}$ is the maximal compact subgroup of $\mathfrak{G}$, with τ the Cartan involution. Note also that any group Lie $\mathfrak{G}$ can be seen as the symmetric space $(\mathfrak{G} \times \mathfrak{G})/\Delta\mathfrak{G}$, where $\Delta\mathfrak{G}$ is $\mathfrak{G}$ acting diagonally on $\mathfrak{G} \times \mathfrak{G}$:

$$g.(g_1, g_2) = (gg_1, gg_2). \tag{4.3}$$

In this case, one recovers principal chiral models, here in a superconformal version.

Although we could work with an arbitrary number of supercharges, we work now with $(1,1)$ supersymmetry. Generalization to other cases follows easily, as we have seen in last section for a simpler model.

So let $\mathcal{V}$ be a superfield with values in $\mathfrak{G}/\mathfrak{H}$, parametrized by superfields ϕ^i. We define the field strength $\mathcal{G} = \mathcal{X} + \mathcal{Y}$ as

$$\begin{aligned} G &= D\mathcal{V}\,\mathcal{V}^{-1} = X + Y \\ \overline{G} &= \overline{D}\mathcal{V}\,\mathcal{V}^{-1} = \overline{X} + \overline{Y}, \end{aligned} \tag{4.4}$$

where X and $\overline{X}$ have values in $\mathfrak{h}$, Y and $\overline{Y}$ in $\mathfrak{h}^{\perp}$.

Because of (4.2), Bianchi identities decomposes on $\mathfrak{h}$ and $\mathfrak{h}^{\perp}$ as

$$\begin{aligned} \overline{D}X + D\overline{X} - \left[X, \overline{X}\right] &= \left[Y, \overline{Y}\right] \\ \overline{\mathcal{D}}Y + \mathcal{D}\overline{Y} &= 0, \end{aligned} \tag{4.5}$$

where $\mathcal{D}$ and $\overline{\mathcal{D}}$ are covariant derivatives with respect to $\mathfrak{H}$:

$$\begin{aligned} \mathcal{D}\overline{Y} &= D\overline{Y} - \left[X, \overline{Y}\right] \\ \overline{\mathcal{D}}Y &= DY - \left[\overline{X}, Y\right]. \end{aligned} \tag{4.6}$$

$\mathcal{G}$ is invariant under global $\mathfrak{G}$ transformations acting on the right and covariant under local $\mathfrak{H}$ on the left:

$$\mathcal{V}(z,\bar{z},\theta,\bar{\theta}) \;\longrightarrow\; \Xi(z,\bar{z},\theta,\bar{\theta})\,\mathcal{V}(z,\bar{z},\theta,\bar{\theta})\,\Lambda. \tag{4.7}$$

Invariance under a global $\Lambda \in \mathfrak{G}$ is the same as above. Gauge action under $\Xi(z, \bar{z}, \theta, \bar{\theta}) \in \mathfrak{H}$ acts on G as

$$D\mathcal{V}\,\mathcal{V}^{-1} \;\longrightarrow\; \Xi\, D\mathcal{V}\,\mathcal{V}^{-1}\,\Xi^{-1} + D\Xi\,\Xi^{-1}. \tag{4.8}$$

From (4.2) and $\Xi \in \mathfrak{H}$ we get in terms of X and Y

$$\begin{aligned} X &\longrightarrow \Xi\, X\, \Xi^{-1} + D\Xi\,\Xi^{-1} \\ Y &\longrightarrow \Xi\, Y\, \Xi^{-1}. \end{aligned} \tag{4.9}$$

For the right components, we get similarly

$$\begin{aligned} \overline{X} &\longrightarrow \Xi\, \overline{X}\, \Xi^{-1} + \overline{D}\Xi\,\Xi^{-1} \\ \overline{Y} &\longrightarrow \Xi\, \overline{Y}\, \Xi^{-1}. \end{aligned} \tag{4.10}$$

We check that $\mathcal{X} = \begin{pmatrix} X \\ \overline{X} \end{pmatrix}$ transforms indeed as a gauge field.

This gauge invariance allow us to parametrized $\mathfrak{G}/\mathfrak{H}$ with a fixed set of representatives in $\mathfrak{G}$. A common choice for $\mathfrak{G}$ a simple group and $\mathfrak{H}$ its maximal compact subgroup is the Borel gauge: $\mathcal{V}$ is taken in the Borel subgroup $\mathfrak{B}$ of $\mathfrak{G}$. The Borel subalgebra $\mathfrak{b}$ is spanned by Cartan elements and generators associated to positive roots. We will use such a gauge choice when we deal with the $\mathrm{SL}(2)/\,\mathrm{SO}(2)$ example. Global $\mathfrak{G}$ transformations do not always preserve the gauge. In the Borel gauge, only elements of the Borel subgroup leave the gauge invariant. Other transformations must be compensated by $\mathfrak{H}$ transformations (Iwasawa decomposition ensures it is always possible), such that

$$\Xi(\mathcal{V}, \Lambda)\, \mathcal{V}\, \Lambda \in \mathfrak{B}. \tag{4.11}$$

4.2 Superconformal selfduality

We want now to get equations of motion by imposing a selfduality constraint. We need an $\mathfrak{H}$-invariant pseudo-involution[4] $\mathcal{S}$ acting on $\mathfrak{g}$:

$$\mathcal{S} = \Xi\, \mathcal{S}\, \Xi^{-1} \tag{4.12}$$

for any $\Xi \in \mathfrak{H}$.

It allows us to write a gauge-invariant selfduality equation

$$*\mathcal{S}\mathcal{Y} = \mathcal{Y}, \tag{4.13}$$

i.e.

$$\begin{aligned} -i\mathcal{S}Y &= Y \\ i\mathcal{S}\overline{Y} &= \overline{Y}. \end{aligned} \tag{4.14}$$

With the second Bianchi identity of (4.5) applied to the field $\mathcal{S}\mathcal{Y}$, we get the equation of motion

$$\overline{\mathcal{D}}Y - \mathcal{D}\overline{Y} = 0. \tag{4.15}$$

This model has several symmetries: superconformal symmetry on the worldsheet, global $\mathfrak{G}$ and local $\mathfrak{H}$ in the target space $\mathcal{M} = \mathfrak{G}/\mathfrak{H}$.

[4] It has square -1 in the Euclidean setting, but, as before, it would be a real involution with square identity in the lorentzian case: $\mathcal{S}^2 *^2 = 1$.

4.3 σ-model action

Though our aim is to describe models where the equations of motion come from a selfduality constraint, it is worth remarking that the equation of motion (4.15) extremizes the superconformal σ-model action [26, 27]

$$S = \frac{1}{2} \int \mathrm{d}z^2 \mathrm{d}\theta^2 \ \mathrm{Tr} \left(\mathcal{Y} \wedge * \mathcal{Y} \right) , \tag{4.16}$$

with the superspace Hodge star $*$ defined above. The trace is computed using an $\mathfrak{h}$ invariant bilinear tensor η_{ab} on $\mathfrak{g}$. (In some cases, there are several inequivalent invariant tensors η.) The wedge product is defined as follows:

$$\mathcal{Y} \wedge \mathcal{Y}' = Y\overline{Y}' - \overline{Y}Y'. \tag{4.17}$$

The equation of motion is indeed

$$\overline{\mathcal{D}}Y - \mathcal{D}\overline{Y} = 0. \tag{4.18}$$

We have showed here that selfdual sets of fields are special solutions of these models. We will prove in section 5 that all solutions to this equation of motion can be seen as selfdual configurations, if one introduces a infinite set of fields.

4.4 Example: SL(2)/ SO(2)

We take $\mathfrak{G} = \mathrm{SL}(2)$, and $\mathfrak{H}$ its maximal compact subgroup $SO(2)$, characterized by the Cartan involution

$$O \longrightarrow {}^tO^{-1}, \tag{4.19}$$

which gives on the tangent algebra

$$\tau(M) = -M. \tag{4.20}$$

We take

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.21}$$

as basis element of $\mathfrak{h} = \mathfrak{so}(2)$ and

$$V^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$V^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.22}$$

as basis of $\mathfrak{h}^\perp$.

We parametrize SL(2)/ SO(2) elements in the Borel gauge of upper triangular matrices as

$$\mathcal{V} = \begin{pmatrix} e^\phi & Ne^\phi \\ 0 & e^{-\phi} \end{pmatrix} \tag{4.23}$$

with real superfields N and ϕ.

The field strength is

$$
\begin{aligned}
G = D\mathcal{V}\,\mathcal{V}^{-1} &= \begin{pmatrix} D\phi & e^{2\phi}DN \\ 0 & -D\phi \end{pmatrix} \\
\overline{G} = \overline{D}\mathcal{V}\,\mathcal{V}^{-1} &= \begin{pmatrix} \overline{D}\phi & e^{2\phi}\overline{D}N \\ 0 & -\overline{D}\phi \end{pmatrix}.
\end{aligned}
\tag{4.24}
$$

It decomposes on $\mathfrak{h}\oplus\mathfrak{h}^{\perp}$ as

$$
\begin{aligned}
G &= \frac{1}{2}e^{2\phi}DN\,U + \frac{1}{2}e^{2\phi}DN\,V^1 + D\phi\,V^2 \\
\overline{G} &= \frac{1}{2}e^{2\phi}\overline{D}N\,U + \frac{1}{2}e^{2\phi}\overline{D}N\,V^1 + \overline{D}\phi\,V^2.
\end{aligned}
\tag{4.25}
$$

A selfduality constraint can be imposed if we have some operator $\mathcal{S}$ acting on $\mathfrak{h}^{\perp}$ with $\mathcal{S}^2=-1$ and which commutes with $\mathfrak{so}(2)$. There is a unique solution, up to a global minus sign:

$$
\begin{aligned}
\mathcal{S}V^1 &= -V^2 \\
\mathcal{S}V^2 &= V^1.
\end{aligned}
\tag{4.26}
$$

The selfduality equation $*\mathcal{S}\mathcal{Y}=\mathcal{Y}$ reads

$$
\begin{aligned}
-i\mathcal{S}\left(\frac{1}{2}e^{2\phi}DN\,V^1 + D\phi\,V^2\right) &= \frac{1}{2}e^{2\phi}DN\,V^1 + D\phi\,V^2 \\
i\mathcal{S}\left(\frac{1}{2}e^{2\phi}\overline{D}N\,V^1 + \overline{D}\phi\,V^2\right) &= \frac{1}{2}e^{2\phi}\overline{D}N\,V^1 + \overline{D}\phi\,V^2
\end{aligned}
\tag{4.27}
$$

i.e.

$$
\begin{aligned}
\frac{i}{2}e^{2\phi}DN &= D\phi \\
-\frac{i}{2}e^{2\phi}\overline{D}N &= \overline{D}\phi.
\end{aligned}
\tag{4.28}
$$

This selfduality constraint is easily solved in terms of chiral and antichiral superfields f and $\tilde{f}$:

$$
\begin{aligned}
e^{-2\phi} &= \frac{1}{2}\left(f(z,\theta)+\tilde{f}(\bar{z},\bar{\theta})\right) \\
N &= -\frac{1}{2i}\left(f(z,\theta)-\tilde{f}(\bar{z},\bar{\theta})\right).
\end{aligned}
\tag{4.29}
$$

The reality of ϕ and N imposes $\tilde{f}=\bar{f}$ and finally the selfdual configurations are expressed in term of a single chiral superfield f as

$$
\begin{aligned}
e^{\phi} &= \frac{1}{\sqrt{\Re(f)}} \\
N &= -\Im(f).
\end{aligned}
\tag{4.30}
$$

As we have seen above, these selfdual configurations are in particular solutions to the equations of motion of the σ-model action (4.16).

5. Loop group symmetry

5.1 Loop group σ-model

We show now how the symmetric space $\mathfrak{G}/\mathfrak{H}$ σ-model can be seen as an infinite-dimensional $\widetilde{\mathfrak{G}}/\widetilde{\mathfrak{H}}$ σ-model with a self-duality contraint, where $\widetilde{\mathfrak{G}}$ is the loop group extension of $\mathfrak{G}$ and $\widetilde{\mathfrak{H}}$ a maximal subgroup. By construction, it will be endorsed with global $\widetilde{\mathfrak{G}}$ symmetry and local $\widetilde{\mathfrak{H}}$ gauge symmetry. We still work with $(1,1)$ supersymmetry, but it is straightforward to do it with a different or vanishing number of supercharges.

The loop extension $\widetilde{\mathfrak{G}}$ of $\mathfrak{G}$ is the infinite-dimensional Lie group which has tangent Lie algebra

$$\widetilde{\mathfrak{g}} = \mathbb{R}\left[t, \frac{1}{t}\right] \otimes \mathfrak{G}, \tag{5.1}$$

the loop algebra extension of $\mathfrak{g}$. In practice, we deal with elements of this group and this algebra as functions of the spectral parameter t into respectively $\mathfrak{G}$ and $\mathfrak{g}$.

Let $\mathfrak{h}$ be a maximal subalgebra of $\mathfrak{g}$, defined by an involution $\tilde{\tau}$. τ is extended on $\widetilde{\mathfrak{g}}$ by[5]

$$\tilde{\tau}(\mathcal{A}(t)) = \tau\left(\mathcal{A}\left(-\frac{1}{t}\right)\right), \tag{5.2}$$

with $\mathcal{A}(t) \in \widetilde{\mathfrak{g}}$ [11, 12, 14]. With an angle parameter α, $t = \tan\left(\frac{\alpha}{2}\right)$, this involution exchanges α and $\alpha + \pi$:

$$\tilde{\tau}(\mathcal{A}(\alpha)) = \tau(\mathcal{A}(\alpha + \pi)). \tag{5.3}$$

We denote by $\widetilde{\mathfrak{h}}$ the subalgebra of fixed points of $\widetilde{\mathfrak{g}}$ by $\tilde{\tau}$. It is not the loop extension of $\widetilde{\mathfrak{h}}$. The involution is defined in the same way on the group $\widetilde{\mathfrak{G}}$ itself, with the subgroup of fixed points $\widetilde{\mathfrak{H}}$.

The component of degree zero in a t expansion of $\widetilde{\mathfrak{G}}$ is the original group $\mathfrak{G}$. As $\tilde{\tau}$ reduces to τ on that component, the $t = 0$ part of $\widetilde{\mathfrak{H}}$ is $\mathfrak{H}$. Of course, the same is true for tangent algebras. Note that $\widetilde{\mathfrak{H}}$ is not the loop extension of $\mathfrak{H}$.

We take a field $\widetilde{\mathcal{V}}$ in the coset symmetric space $\widetilde{\mathfrak{G}}/\widetilde{\mathfrak{H}}$. Due to the gauge freedom, we choose representatives in a "Borel gauge": $\widetilde{\mathcal{V}}(t)$ must be analytic inside a unit disc around zero [11, 13, 14, 23]. (Note that the choice of this point is arbitrary. It will allow to recover physical quantities as the $t = 0$ value of fields.) The existence and uniqueness of such a gauge is a Riemann-Hilbert problem: if we have $\widetilde{\mathcal{V}} \in \widetilde{\mathfrak{G}}$, we can decompose $\widetilde{\mathcal{V}}\,\tilde{\tau}\left(\widetilde{\mathcal{V}}^{-1}\right)$ as $\widetilde{\mathcal{V}}_+\,\tilde{\tau}\left(\widetilde{\mathcal{V}}_-^{-1}\right)$, with $\widetilde{\mathcal{V}}_+$ analytic inside the unit disc and $\widetilde{\mathcal{V}}_-$ analytic outside. (t lives on the Riemann sphere.) We are left with gauge freedom at $t = 0$, that we fix as in the $\mathfrak{G}/\mathfrak{H}$ σ-model: we set $\widetilde{\mathcal{V}}(0)$ to be in a given Borel subgroup of $\mathfrak{G}$.

The field strength is[6]

$$\widetilde{\mathcal{G}} = \begin{pmatrix} \widetilde{G} \\ \widetilde{\overline{G}} \end{pmatrix} = \begin{pmatrix} D\widetilde{\mathcal{V}}\,\widetilde{\mathcal{V}}^{-1} \\ \overline{D}\widetilde{\mathcal{V}}\,\widetilde{\mathcal{V}}^{-1} \end{pmatrix}. \tag{5.4}$$

[5] This is for Euclidean signature. For a lorentzian worldsheet, the minus sign would not be there.

[6] D and $\overline{D}$ derivatives acts on superspace variables z, $\bar{z}$, θ and $\bar{\theta}$, not on the spectral parameter t.

A Bianchi identity follows naturally from the definition:

$$D\widetilde{\widetilde{G}} + \overline{D}\widetilde{G} = \left[\widetilde{G}, \widetilde{\widetilde{G}}\right]. \tag{5.5}$$

If $\widetilde{\mathcal{V}}$ is in the Borel gauge defined above, $\widetilde{\mathcal{G}}$ has an expansion

$$\widetilde{\mathcal{G}} = \mathcal{X} + \mathcal{Y} + \sum_{n>0} \mathcal{A}_n t^n, \tag{5.6}$$

with

$$\begin{aligned} \mathcal{X} &\in \mathfrak{h} \\ \mathcal{Y} &\in \mathfrak{h}^\perp \\ \mathcal{A}_n &\in \mathfrak{g}. \end{aligned} \tag{5.7}$$

The decomposition $\widetilde{\mathcal{G}} = \widetilde{\mathcal{X}} + \widetilde{\mathcal{Y}}$ on $\widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{h}}^\perp$ is

$$\begin{aligned} \widetilde{\mathcal{X}} &= \mathcal{X} + \frac{1}{2}\sum_{n>0} \mathcal{A}_n t^n + \frac{1}{2}\sum_{n>0} \tau(\mathcal{A}_n)\left(-\frac{1}{t}\right)^n \\ \widetilde{\mathcal{Y}} &= \mathcal{Y} + \frac{1}{2}\sum_{n>0} \mathcal{A}_n t^n - \frac{1}{2}\sum_{n>0} \tau(\mathcal{A}_n)\left(-\frac{1}{t}\right)^n . \end{aligned} \tag{5.8}$$

5.2 Selfduality constraint

In order to impose a selfduality constraint, we define the operator $\mathcal{S}$ in the following way:

$$\mathcal{S} : \mathcal{A}(t) \;\longrightarrow\; -t\,\tilde{\tau}(\mathcal{A}(t)). \tag{5.9}$$

It is not hard to check that $\mathcal{S}$ is a pseudo-involution[7]:

$$\mathcal{S}^2 = -1, \tag{5.10}$$

such that the full duality operator $*\mathcal{S}$ is a real involution:

$$(*\mathcal{S})^2 = 1. \tag{5.11}$$

As in the finite-dimensional case, we impose the selfduality constraint

$$*\mathcal{S}\widetilde{\mathcal{Y}} = \widetilde{\mathcal{Y}}, \tag{5.12}$$

where $*$ is here the superspace Hodge star defined in section 3.1, but in a purely bosonic case would be the usual Hodge operator.

For an element $\widetilde{\mathcal{Y}}$ of $\widetilde{\mathfrak{h}}^\perp$, which by definition satisfies

$$\tilde{\tau}(\widetilde{\mathcal{Y}}) = -\widetilde{\mathcal{Y}}, \tag{5.13}$$

we have

$$\mathcal{S}\widetilde{\mathcal{Y}}(t) = t\widetilde{\mathcal{Y}}(t). \tag{5.14}$$

[7]In a lorentzian signature, it would square to $+1$.

JHEP10(2004)071

The selfduality constraint (5.12) is thus equivalent to

$$*\mathcal{T}\widetilde{\mathcal{Y}} = \widetilde{\mathcal{Y}}, \tag{5.15}$$

where $\mathcal{T}$ is the operator shifting elements of $\widetilde{\mathfrak{g}}$ by one degree in t:

$$\mathcal{T} : \mathcal{A}(t) \;\longrightarrow\; t\,\mathcal{A}(t). \tag{5.16}$$

In components of the t-expansion of $\widetilde{\mathcal{Y}}$, the selfduality equation gives

$$\begin{aligned} \mathcal{A}_{2p} &= 2(-1)^p\,\mathcal{Y} \\ \mathcal{A}_{2p+1} &= 2(-1)^p *\mathcal{Y}. \end{aligned} \tag{5.17}$$

If we sum up all components, we get for $\widetilde{\mathcal{G}}$

$$\widetilde{\mathcal{G}} = \mathcal{X} + \frac{1-t^2}{1+t^2}\,\mathcal{Y} + \frac{2t}{1+t^2} *\mathcal{Y}, \tag{5.18}$$

which is well known as the Lax pair equation[8] associated to the $\mathfrak{G}/\mathfrak{H}$ σ-model described in section 4.3 [28, 29]. In term of the angle variable α, it reads

$$\widetilde{\mathcal{G}} = \mathcal{X} + \cos(\alpha)\,\mathcal{Y} + \sin(\alpha) *\mathcal{Y}. \tag{5.19}$$

In superconformal coordinates, with covariant derivatives

$$\begin{aligned} \mathcal{D}\widetilde{\mathcal{V}} &= D\widetilde{\mathcal{V}} - X\widetilde{\mathcal{V}} \\ \overline{\mathcal{D}}\widetilde{\mathcal{V}} &= \overline{D}\widetilde{\mathcal{V}} - \overline{X}\widetilde{\mathcal{V}}, \end{aligned} \tag{5.20}$$

we have

$$\begin{aligned} \mathcal{D}\widetilde{\mathcal{V}}\,\widetilde{\mathcal{V}}^{-1} &= \frac{i+t}{i-t}\,Y \\ \overline{\mathcal{D}}\widetilde{\mathcal{V}}\,\widetilde{\mathcal{V}}^{-1} &= \frac{i-t}{i+t}\,\overline{Y}. \end{aligned} \tag{5.21}$$

It is a linear system which allows to get $\widetilde{\mathcal{V}}$ by integration, at least formally.

The Bianchi identity and the equation of motion of the original field

$$\mathcal{V} = \widetilde{\mathcal{V}}(0) \tag{5.22}$$

follow from the compatibility condition, or Maurer-Cartan equation, of this linear system. The Bianchi identity of $\mathcal{V}$ comes simply from the $t = 0$ (i.e. $\alpha = 0$) part of (5.5). Indeed, the truncation of (5.21) to $t = 0$ is the defining equation of Y and $\overline{Y}$ in term of $\mathcal{V}$, and we find back Bianchi identities (4.5).

The equation of motion comes from the Bianchi identity of $\widetilde{\mathcal{G}}$ at $\theta = \frac{\pi}{2}$ $(t = 1)$:

$$\mathcal{D}\overline{Y} - \overline{\mathcal{D}}Y = 0. \tag{5.23}$$

[8]In lorentzian signature, it would be $\widetilde{\mathcal{G}} = \mathcal{X} + \frac{1+t^2}{1-t^2}\,\mathcal{Y} + \frac{2t}{1-t^2} *\mathcal{Y}$.

5.3 Symmetries

By construction, this model has global $\widetilde{\mathfrak{G}}$ symmetry and local $\widetilde{\mathfrak{H}}$ gauge symmetry, because $\mathcal{S}$ commutes with $\widetilde{\mathfrak{H}}$. Indeed, if we act on $\widetilde{\mathcal{V}}$ with a global $\widetilde{\Lambda} \in \widetilde{\mathfrak{G}}$, we may have to compensate by an $\widetilde{\mathfrak{H}}$ gauge transformation $\tilde{\tilde{\Xi}}$ to remain in the Borel gauge:

$$\widetilde{\mathcal{V}} \;\longrightarrow\; \tilde{\tilde{\Xi}}\,\widetilde{\mathcal{V}}\,\widetilde{\Lambda}. \tag{5.24}$$

Under such a transformation, $\widetilde{\mathcal{Y}}$ becomes

$$\widetilde{\mathcal{Y}} \;\longrightarrow\; \tilde{\tilde{\Xi}}\,\widetilde{\mathcal{Y}}\,\tilde{\tilde{\Xi}}^{-1}. \tag{5.25}$$

The constraint (5.15), equivalent to the selfduality constraint (5.12), is invariant provided

$$\tilde{\tilde{\Xi}}\,\mathcal{T}\,\tilde{\tilde{\Xi}}^{-1} = \mathcal{T}, \tag{5.26}$$

which is indeed verified. Under the compensating $\widetilde{\mathfrak{H}}$ gauge transformation, $\widetilde{\mathcal{X}}$ varies as a gauge field:

$$\begin{aligned} \widetilde{X} &\longrightarrow \tilde{\tilde{\Xi}}\,\widetilde{X}\,\tilde{\tilde{\Xi}}^{-1} + D\tilde{\tilde{\Xi}}\,\tilde{\tilde{\Xi}}^{-1} \\ \widetilde{\overline{X}} &\longrightarrow \tilde{\tilde{\Xi}}\,\widetilde{\overline{X}}\,\tilde{\tilde{\Xi}}^{-1} + \overline{D}\tilde{\tilde{\Xi}}\,\tilde{\tilde{\Xi}}^{-1}. \end{aligned} \tag{5.27}$$

One recovers transformation of physical field strength $\mathcal{X}$ and $\mathcal{Y}$ by taking the $t = 0$ component of transfomration laws of $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{Y}}$.

$\widetilde{\mathfrak{G}}$ transformations $g\,t^n$ with $n > 0$ have no effect on the physical field $\mathcal{V}$: they do not require gauge compensation and modify only integration constants of higher levels of $\widetilde{\mathcal{V}}$ [14]. For $n = 0$, we recover the usual symmetry of the $\mathfrak{G}/\mathfrak{H}$ σ-model. For $n < 0$, physical fields are modified by gauge compensation $\tilde{\tilde{\Xi}}$. In fact, these non-local additional symmetries can be used to generate new (classical) solutions of the model [32]–[34]. The fact that some of the transformations are hidden whereas others are visible through the gauge compensation is due to the fixing of the gauge that we use to extract physical fields. If we restore gauge freedom, we cannot get rid of the full group of symmetry $\widetilde{\mathfrak{G}}$. It should be noted that, if the rigid symmetry group is the same as in the bosonic case, in supersymmetric models the gauge symmetry is enlarged: gauge transformations depend on both bosonic and fermionic coordinates.

Acknowledgments

We thank B. Julia for discussions on related topics, and also A. Keurentjes for comments. This work is supported in part by the "Actions de Recherche Concertées" of the "Direction de la Recherche Scientifique - Communauté Française de Belgique", by a "Pôle d'Attraction Interuniversitaire" (Belgium) and by IISN-Belgium (convention 4.4505.86). Support from the European Commission RTN programme HPRN-CT-00131, in which we are associated to K.U. Leuven, is also acknowledged.

JHEP10(2004)071

References

[1] E. Cremmer, B. Julia, H. Lu and C.N. Pope, *Dualisation of dualities, 2. Twisted self-duality of doubled fields and superdualities*, *Nucl. Phys.* **B 535** (1998) 242 [hep-th/9806106].

[2] P. Henry-Labordere, B. Julia and L. Paulot, *Borcherds symmetries in M-theory*, *J. High Energy Phys.* **04** (2002) 049 [hep-th/0203070].

[3] P. Henry-Labordere, B. Julia and L. Paulot, *Real borcherds superalgebras and M-theory*, *J. High Energy Phys.* **04** (2003) 060 [hep-th/0212346].

[4] P.C. West, *E_{11} and M-theory*, *Class. and Quant. Grav.* **18** (2001) 4443 [hep-th/0104081].

[5] I. Schnakenburg and P.C. West, *Kac-Moody symmetries of IIB supergravity*, *Phys. Lett.* **B 517** (2001) 421 [hep-th/0107181].

[6] I. Schnakenburg and P.C. West, *Massive IIA supergravity as a non-linear realisation*, *Phys. Lett.* **B 540** (2002) 137 [hep-th/0204207].

[7] E. Cremmer, B. Julia, H. Lu and C.N. Pope, *Higher-dimensional origin of $D = 3$ coset symmetries*, hep-th/9909099.

[8] Y. Dolivet, P. Henry-Labordère, B. Julia, L. Paulot, *Superalgebras of oxidation chains*, hep-th/0405xxx.

[9] F. Englert, L. Houart, A. Taormina and P. West, *The symmetry of M3-theories*, *J. High Energy Phys.* **09** (2003) 020 [hep-th/0304206].

[10] B.L. Julia, *Dualities in the classical supergravity limits: dualisations, dualities and a detour via 4K+2 dimensions*, hep-th/9805083.

[11] P. Breitenlohner, D. Maison, *Explicit and hidden symmetries of dimensionally reduced (super)gravity theories*, in *Retzbach 1983, proceedings, solutions of Einstein's equations: techniques and results*, C. Hoensealers and W. Dietz eds., Springer, 1984, p. 276 MPI-PAE/PTh 1/84.

[12] B. Julia, *Infinite Lie algebras in physics*, in *Johns Hopkins workshop on current problems in particle theory*, Baltimore, 1981.

[13] H. Nicolai, *Two-dimensional gravities and supergravities as integrable system*, in *Schladming 1991, proceedings, recent aspects of quantum fields*, H. Mitter and H. Gausterer eds., Springer, 1991, p. 231.

[14] B. Julia and H. Nicolai, *Conformal internal symmetry of 2D sigma-models coupled to gravity and a dilaton*, *Nucl. Phys.* **B 482** (1996) 431 [hep-th/9608082].

[15] D. Bernard and B. Julia, *Twisted self-duality of dimensionally reduced gravity and vertex operators*, *Nucl. Phys.* **B 547** (1999) 427 [hep-th/9712254].

[16] H. Nicolai, *The integrability of N=16 supergravity*, *Phys. Lett.* **B 194** (1987) 402.

[17] H. Nicolai and N.P. Warner, *The structure of N=16 supergravity in two-dimensions*, *Commun. Math. Phys.* **125** (1989) 369.

[18] H. Nicolai, *New linear systems for 2D Poincaré supergravities*, *Nucl. Phys.* **B 414** (1994) 299.

[19] H. Nicolai and H. Samtleben, *Integrability and canonical structure of $D = 2$, $N = 16$ supergravity*, *Nucl. Phys.* **B 533** (1998) 210 [hep-th/9804152].

JHEP10(2004)071

[20] D. Korotkin and H. Nicolai, *Isomonodromic quantization of dimensionally reduced gravity*, *Nucl. Phys.* **B 475** (1996) 397 [hep-th/9605144].

[21] D. Korotkin and H. Samtleben, *Quantization of coset space sigma-models coupled to two-dimensional gravity*, *Commun. Math. Phys.* **190** (1997) 411 [hep-th/9607095].

[22] D. Korotkin and H. Samtleben, *Yangian symmetry in integrable quantum gravity*, *Nucl. Phys.* **B 527** (1998) 657 [hep-th/9710210].

[23] H. Nicolai, D. Korotkin and H. Samtleben, *Integrable classical and quantum gravity*, hep-th/9612065.

[24] A.J. Nurmagambetov, *The sigma-model representation for the duality-symmetric $D = 11$ supergravity*, hep-th/0312157.

[25] A.J. Nurmagambetov, *On the sigma-model structure of type-IIA supergravity action in doubled field approach*, *Sov. Phys. JETP Lett.* **79** (2004) 191 [hep-th/0403100].

[26] P. Di Vecchia and S. Ferrara, *Classical solutions in two-dimensional supersymmetric field theories*, *Nucl. Phys.* **B 130** (1977) 93.

[27] E. Witten, *A supersymmetric form of the nonlinear sigma model in two-dimensions*, *Phys. Rev.* **D 16** (1977) 2991.

[28] S. Ferrara, L. Girardello and S. Sciuto, *An infinite set of conservation laws of the supersymmetric sine-Gordon theory*, *Phys. Lett.* **B 76** (1978) 303.

[29] J. Hruby, *On the supersymmetric sine-Gordon model and a two-dimensional 'bag'*, *Nucl. Phys.* **B 131** (1977) 275.

[30] T.L. Curtright and C.K. Zachos, *Nonlocal currents for supersymmetric nonlinear models*, *Phys. Rev.* **D 21** (1980) 411.

[31] L.L. Chau, M.L. Ge and Y.S. Wu, *Noether currents and algebraic structure of the hidden symmetry for superchiral fields*, *Phys. Rev.* **D 25** (1982) 1080.

[32] L. Dolan, *Kac-Moody algebra is hidden symmetry of chiral models*, *Phys. Rev. Lett.* **47** (1981) 1371.

[33] H. Eichenherr and M. Forger, *Higher local conservation laws for nonlinear sigma models on symmetric spaces*, *Commun. Math. Phys.* **82** (1981) 227.

[34] Y.S. Wu, *Extension of the hidden symmetry algebra in classical principal chiral models*, *Nucl. Phys.* **B 211** (1983) 160.

[35] M. Jacques and Y. Saint-Aubin, *Infinite dimensional Lie algebras acting on the solution space of various sigma models*, *J. Math. Phys.* **28** (1987) 2463.

[36] J.H. Schwarz, *Classical symmetries of some two-dimensional models coupled to gravity*, *Nucl. Phys.* **B 454** (1995) 427 [hep-th/9506076].

[37] J.M. Evans, D. Kagan and C.A.S. Young, *Non-local charges and quantum integrability of sigma models on the symmetric spaces* $\mathrm{SO}(2N)/\mathrm{SO}(N) \times \mathrm{SO}(N)$ *and* $\mathrm{Sp}(2N)/\mathrm{Sp}(N) \times \mathrm{Sp}(N)$, *Phys. Lett.* **B 597** (2004) 112 [hep-th/0404003].